Tutorial-7 B.Tech. Semester-I

Section A : Improper integrals, Beta & Gama function, and their properties

Q1: Determine if each of the following integrals converge or diverge. If integral converges Determine its value

1) $\int_0^{\infty}(1+2x)\,e^{-x}dx$ 2. $\int_{-\infty}^{0}(1+2x)\,e^{-x}dx$ [Hint: u =1+2x, v = $-e^{-x}$]

3) $\int_{-5}^{1}\frac{1}{10+2x}dx$ Hint: $\int\frac{1}{10+2x}dx=\frac{1}{2}ln(10+2x)+c$

4) $\int_1^2\frac{4x}{\sqrt[3]{x^2-4}}dx$ Hint: $\int\frac{4x}{\sqrt[3]{x^2-4}}dx=3(x^2-4)^{\frac{2}{3}}+c$

5) $\int_{-\infty}^{1}\sqrt{6-x}\,dx$ Hint: $\int\sqrt{6-x}\,dx=-\frac{2}{3}(6-x)^{\frac{3}{2}}+c$

6) $\int_2^{\infty}\frac{9}{(1-3x)^4}dx$ Hint: $\int\frac{9}{(1-3x)^4}dx=\frac{1}{(1-3x)^3}+c$

7) $\int_0^4\frac{x}{x^2-9}dx$ Hint: $\int\frac{x}{x^2-9}dx=\frac{1}{2}ln(x^2-9)+c$

8) $\int_{-\infty}^{\infty}\frac{6x^3}{(x^4+1)^2}dx$ Hint: $\int\frac{6x^3}{(x^4+1)^2}dx=-\frac{3}{2}\frac{1}{(x^4+1)}+c$

9) $\int_1^4\frac{1}{x^2+x-6}dx$ Hint: $\int\frac{1}{(x+3)\ (x-2)}dx=\frac{1}{5}ln(x-2)-\frac{1}{5}ln(x+3)+c$

10) $\int_{-\infty}^{0}\frac{e^{\frac{1}{x}}}{x^2}dx$ Hint: $\int\frac{e^{\frac{1}{x}}}{x^2}dx=-e^{\frac{1}{x}}+c$

Answers: 1. CONVERGES, VALUE = 3 2, 3 . DIVERGES 4. CONVERGES, VALUE = $(3)^{5/3}$ 5. DIVERGES 6. CONVERGES, VALUE = 1/125 7. DIVERGES 8. CONVERGES, VALUE = 0 9. CONVERGES, VALUE = 1 10. CONVERGES, VALUE = 1

Beta & Gama function, and their properties

11) Evaluate $\int_0^1\frac{dx}{\sqrt{1-x^4}}$ ANS: $\frac{\sqrt{\pi}}{4}\frac{\Gamma(1/2)}{\Gamma(3/4)}$

12) Prove that $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$. [Hint: In $\frac{\left|\frac{p+1}{2}\right|\left|\frac{q+1}{2}\right|}{\left|\frac{p+q+2}{2}\right|}=2\int_0^{\pi/2}\mathrm{Sin}^p\theta\,\mathrm{Cos}^q\theta d\theta$, put $p=q=0$]

13) $Given that \int_0^{\infty}\frac{x^{n-1}}{1+x}dx=\frac{\pi}{sin\,n\pi}, then show that \Gamma(n)\Gamma(1-n)=\frac{\pi}{sin(n\pi)}$

14) Show that $\overline{|m}\ \overline{\left|m+\frac{1}{2}}=\frac{\sqrt{\pi}}{2^{2m-1}}\overline{|2m}$ where *m* is the integer.

15) Evaluate in term of Beta function.

a) $\int_0^1\frac{x}{\sqrt{1-x^2}}dx$ [Hint: Put $x^2=y$] ANS: $\frac{1}{2}\beta\left(1,\frac{1}{2}\right)$

b) $\int_0^3\frac{dx}{\sqrt{9-x^2}}$ [Hint: Put $x^2=9y$] ANS: $\frac{1}{2}\beta\left(\frac{1}{2},\frac{1}{2}\right)$

c) $\int_0^1\frac{xdx}{\sqrt{1-x^5}}$ [Hint: Put $x^5=y$] ANS: $\frac{1}{5}\beta\left(\frac{2}{5},\frac{1}{2}\right)$

d) $\int_0^1\frac{x^2dx}{\sqrt{1-x^5}}$ [Hint: Put $x^5=y$] ANS: $\frac{1}{5}\beta\left(\frac{3}{5},\frac{1}{2}\right)$

16) Evaluate

a) $\int_0^{\infty}e^{-x^2}\,dx$ [Hint: $\overline{|n}=\int_0^{\infty}x^{n-1}e^{-x}dx$ take n =$\frac{1}{2}$] ANS: $\frac{\sqrt{\pi}}{2}$

b) $\int_0^1x^4(1-x)^2\,dx$ [Hint: $\int_0^1x^4(1-x)^2\,dx=\beta(5,3)$] ANS: $\frac{1}{105}$

17) Evaluate

a) $\int_0^1x(8-x^3)^{1/3}\,dx$ [Hint: Put $x^3=8y$] ANS: $\frac{16}{9\sqrt{3}}$

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b) $\int_0^{\frac{\pi}{2}} sin^5\,\theta\, cos^{7/2}\quad \theta d\theta$ [Hint: put 2m-1 =5 and 2n-1 = $\frac{7}{2}$] ANS: $\frac{64}{1989}$

18) Prove that

$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, m>0, n>0$ [Hint: $B(l,m) = \int_0^1 x^{l-1}(1-x)^{m-1}dx$

19) Show that $\beta(m,n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx.$

[Hint: $B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}dx + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx$]

Prove that (a) $\int_0^\infty e^{-ky}y^{n-1}dy = \frac{\overline{|n}}{k^n}$

[Hint: $\overline{|n} = \int_0^\infty x^{n-1}e^{-x}dx$, replace *x* by *ky*, so that *dx = k dy*, hence $\overline{|n}$ = $\int_0^\infty (ky)^{n-1}e^{-ky}.k.dy$

$\overline{|n} = k^n \int_0^\infty e^{-k}\ y^{n-1}dy$]

(b) $\left|\overline{\frac{1}{2}} = \sqrt{\pi}\right.$ [Hint: $\overline{|n} = \int_0^\infty x^{n-1}e^{-x}dx$,replace x^n by *y*, $nx^{n-1}\,dx = dy$

then $\overline{|n} = \int_0^\infty y^{\frac{n-1}{n}}e^{-y}\frac{1}{n}\frac{dy}{nx^{n-1}}$

$= \int_0^\infty y^{\frac{n-1}{n}}e^{-y}\frac{1}{n}\frac{dy}{nx^{\frac{n-1}{n}}} = \frac{1}{n}\int_0^\infty e^{-y}\frac{1}{n}dy$ when $n=\frac{1}{2}$, $\left|\overline{\frac{1}{2}} = \frac{1}{\frac{1}{2}}\int_0^\infty e^{-y^2}dy = 2\left[\frac{1}{2}\sqrt{\pi}\right]\right.$]

(c) $\int_0^\infty \left(log\frac{1}{y}\right)^{n-1}dy = \overline{|n}$

[Hint: $\overline{|n} = \int_0^\infty x^{n-1}e^{-x}dx$, substitute e^{-x} by *y*, $-e^{-x}\,dx = dy$,- *x = log y, x = log 1/y.* Then

$\overline{|n} = -\int_1^0 \left(log\frac{1}{y}\right)^{n-1} y.dy$]

(d) $\int_0^{\pi/2} Sin^p\,\theta\, Cos^q\,\theta = \frac{\left(\frac{p+1}{2}\right)\left(\frac{q+1}{2}\right)}{2\left(\frac{p+q+2}{2}\right)}$

[Hint: $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$, put $x = Sin^2\,\theta$, $dx = 2\,Sin\,\theta\,Cos\,\theta\,d\theta$ and $1-x = 1-Sin^2\,\theta = Cos^2\,\theta$then $B(m,n) = \int_0^{\pi/2} \mathrm{Sin}^{2m-2}\theta\,\mathrm{Cos}^{2n-2}\theta 2\,\mathrm{Sin}\,\theta\,\mathrm{Cos}\,\theta d\theta$ }

Section B (Double integrals and their applications)

1. Find the area of the loop of the curve $\boldsymbol{x^3 + y^3 = 3axy}$. Also find the area bounded between the curve and its asymptote. **Ans:** $\frac{3a^2}{2}$ **and** $\frac{3a^2}{2}$**.**

2. Evaluate $\iint_A (x^2 + y^2)\,dxdy$ over the area A enclosed by the curves $y = 4x, x + y = 3, y = 0$ and $y = 2$. **Ans:** $9\frac{31}{48}$

3. Evaluate the following double integrals

(i) $\int_0^2 \int_0^{x^2} e^{\frac{y}{x}}\,dydx$ **(ii)** $\int_0^{2a} \int_0^{\sqrt{2ax-x^2}} x^2\,dydx$

(iii) $\int_0^a \int_y^a \frac{x^2}{\sqrt{x^2+y^2}}dydx$

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(iv) $\int_1^0 \int_0^1 (x+y)\,dydx$ **(v)** $\int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\,dydx$

Ans: (i) e^2-1 **.(ii)**$\frac{5\pi a^4}{8}$ **(iii)**$\frac{a^3}{3}\log(\sqrt{2}+1)$ **(iv)-1 (v)** $\frac{\pi a^3}{6}$

4. Show that $\int_1^2 \int_3^4 (xy+e^y)\,dxdy = \int_3^4 \int_1^2 (xy+e^y)\,dydx$

5. Show that $\int_1^2 \int_0^{\frac{y}{2}} y\,dy\,dx = \int_1^2 \int_0^{\frac{x}{2}} x\,dx\,dy$

6. Show that $\int_0^1 \int_0^1 \frac{1}{\sqrt{\{1-x^2\}\{1-y^2\}}}\,dy\,dx = \int_0^1 \int_0^1 \frac{1}{\sqrt{\{1-x^2\}\{1-y^2\}}}\,dxdy$

7. Show that $\int_0^1 dx \int_0^1 \frac{x-y}{(x+y)^3}\,dy \neq \int_0^1 dy \int_0^1 \frac{x-y}{(x+y)^3}\,dx$, also find the values of two integrals.

Ans: $L.H.S = \frac{1}{2}$, $R.H.S = -\frac{1}{2}$ Give a conclusion on the basis of the results in Q. 4 to 7.

8. Evaluate $\iint (x^2y^2)\,dxdy$ over the region $x^2+y^2 \leq 1$. **Ans:** $\frac{\pi}{24}$

9. Evaluate $\iint_A (x^2y^2)\,dxdy$ over the region A bounded by the curves $x=0$, $y=0$, **and** $x^2+y^2=1$. **Ans:** $\frac{\pi}{96}$

10. Evaluate $\iint r^2\,d\theta dr$ over the area of the circle $r = a\,cos\theta$ **Ans:** $\frac{4a^3}{9}$

11. Find by double integration the area lying inside the circle $r = a\,sin\theta$ and outside the parabola $r(1+cos\theta)=a$ **Ans:** $\frac{(9\pi+16)}{12}$

12. Change the order of the following double integrations:

(i) $\int_1^\infty \int_1^x xe^{\frac{-x^2}{y}}\,dydx$ **Ans:** $\int_1^\infty \int_y^\infty xe^{\frac{-x^2}{y}}\,dxdy$

(ii) $\int_0^{2a} \int_0^{\sqrt{2ax-x^2}} f(x,y)\,dydx$ **Ans:** $\int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} f(x,y)\,dxdy$

(iii) $\int_0^{acos\alpha} \int_{xtan\alpha}^{\sqrt{a^2-x^2}} f(x,y)\,dy\,dx$ **Ans:** $\int_0^{asin\alpha} \int_0^{ycot\alpha} f(x,y)\,dxdy + \int_{asin\alpha}^a \int_0^{\sqrt{a^2-y^2}} f(x,y)\,dx\,dy$

(iv) $\int_0^a \int_{mx}^{lx} f(x,y)\,dydx$ **Ans:** $\int_0^{am} \int_{\frac{y}{l}}^{\frac{y}{m}} f(x,y)\,dxdy + \int_{am}^{al} \int_{\frac{y}{l}}^a f(x,y)\,dx\,dy$

(v) $\int_0^a \int_{\frac{x^2}{a}}^{2a-x} xy\,dydx$ **Ans:** $\int_0^a \int_0^{\sqrt{ay}} xy\,dxdy + \int_a^{2a} \int_0^{2a-y} xy\,dx\,dy$

13. Express as single integral and evaluate: $\int_0^{\frac{a}{\sqrt{2}}} \int_0^x x\,dxdy + \int_{\frac{a}{\sqrt{2}}}^a \int_0^{\sqrt{a^2-x^2}} x\,dxdy$

Ans: $\int_0^{\frac{a}{\sqrt{2}}} dy \int_y^{\sqrt{a^2-y^2}} x\,dx$; $\frac{5a^3}{6\sqrt{2}}$

14. Convert into polar co-ordinates $\int_0^{2a} \int_0^{2ax-x^2} dydx$ **Ans:** $\int_0^{\frac{\pi}{2}} \int_0^{2acos\theta} r\,d\theta\,dr$

15. Using transformation $x+y=u, y=vu$ show that $\int_0^1 \int_0^{1-x} e^{\frac{y}{x+y}}\,dydx = \frac{e-1}{2}$

16. Using transformation $x-y=u, x+y=v$ show that $\iint_R \cos\frac{x-y}{x+y}\,dxdy = \frac{sin1}{2}$ Where R is the region bounded by $x=0, y=0, x+y=1$.

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17. Find the whole area of the curve $\mathbf{a^2x^2 = y^3(2a - y)}$ by double integration. **Ans:** πa^2

18. Find the area enclosed by the curve $\mathbf{r = 3 + 2cos\theta}$ by double integration. **Ans:** 11π

19. Find the volume of the torus generated by revolving the circle $\mathbf{x^2 + y^2 = 4}$ about the line $x = 3$. **Ans:** $24\pi^2$

20. Find the Center of gravity of the area bounded by the parabola $\mathbf{y^2 = x}$ and the line $\boldsymbol{x + y = 2}$. **Ans:** $(\frac{8}{5}, -\frac{1}{2})$

21. Find the Center of gravity of the loop of the curve $\boldsymbol{r^2 = a^2\, cos2\theta}$. **Ans:** $(\frac{\pi a\sqrt{2}}{8}, 0)$

22. Find the Center of gravity of an arc of the curve $\boldsymbol{x = a(\theta + sin\theta)}$, $\boldsymbol{y = a(1 - cos\theta)}$ in the positive quadrant. **Ans:** $[a(\pi - \frac{4}{3}, \frac{2a}{3}]$

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Tutorial-5 B.Tech. Semester-I

Error, Approximations and Maxima and Minima

1. Compute an approximate value of $(1.04)^{3.01}$ **Ans: 1.12**

2. If $u = x^2y^3/z$, find the maximum percentage error in u if the percentage error in x, y, z are $2, 3$ and 4 respectively. **Ans: 17**

3. If the sides and angles of a plane triangle vary in such a way that its circum radius remains constant prove that $\frac{da}{cosA} + \frac{db}{cosB} + \frac{dc}{cosC} = 0$ where da, db and dc are small increments in the sides a, b and c respectively.

4. Find the shortest distance from origin to the surface $xyz^2 = 2$. Ans: 2

5. Find the dimensions of a rectangular box, with open top and given volume V, so that the total surface area of the box is a minimum. **Ans**: Length=breadth=2 height=$(2V)^{(13)}$

6. Find the Minimum value of $x^2 + y^2 + z^2$ subject to $1/x + 1/y + 1/z = 1$

7. Find the absolute extrema of $f = 3x^2 + 2y^2 - 4y$ in the region bounded by $y = x^2$ and $y = 4$. **Ans**: Abs max= 28 at $(\pm 2, 4)$; Abs. min= -2 at $(0, 1)$

8. Find Min $f = x^2 + y^2$ subject to $x + y = 10$. Ans: 50 at $(5, 5)$

9. Find extremum of $f = x + y + z$ subject to $x^2 + y^2 + z^2 = 12$. **Ans**: Max $f(2, 2, 2) = 6$, Min $f(-2, -2, -2) = -6$.

10. Find extrema of $u = a^2x^2 + b^2y^2 + c^2z^2$ subject to $x^2 + y^2 + z^2 = 1$ and $lx + my + nz = 0$. **Ans** : The extremum can be obtained by the equation $\frac{l^2}{u-a^2} + \frac{m^2}{u-b^2} + \frac{n^2}{u-c^2} = 0$.

11. Find the extrema of $u = x^2+y^2+z^2$ where $ax^2+by^2+cz^2+2fyz+2gzx+2hxy = 1$. **Ans** : The roots of .

12. Find the extreme values of $f(x, y, z) = 2x + 3y + z$ subject to $x^2 + y^2 = 1$ and $x + z = 1$. **Ans with Hint**: for $\lambda_2 = -1, \lambda_1 = \frac{1}{\sqrt{2}}, f = 1 - 5\sqrt{2}$ while for $\lambda_2 = -1, \lambda_1 = -\frac{1}{\sqrt{2}}, f = 1 + 5\sqrt{2}$

Find the local extrema for the following functions:

13. $f = -22x^2 + 22xy - 11y^2 + 110x - 44y - 23$ **Ans**: $f = 120$ is local min at $(3, 1)$

14. $f = x^3 + y^3 - 3xy$ **Ans**: $f = -1$ is local min at $(1, 1)$; $(0, 0)$ is saddle point.

15. $f = xy + \frac{9}{x} + \frac{3}{y}$ **Ans**: $f = 9$ is local min at $(3, 1)$.

Tutorial-06, B. Tech Sem-I 17/11/2025
Jacobians

1. If $u = x + y + z + t, v = x + y - z - t, w = xy - zt, r = x^2 + y^2 - z^2 - t^2$ show that $\frac{\partial(u,v.w,r)}{\partial(x,y,z,t)} =$ 0 and hence find a relation between u, v, w and r. Ans: $uv = r + 2w$

2. Prove that the following functions are not independent. Find the relation between them

(i) $u = x + y + z, v = xy + yz + zx, w = x^3 + y^3 + z^3 - 3xyz$. Ans: $u^3 = 3uv + w$

(ii) $u = x^2 + y^2 + z^2$,$v = x + y + z$,$w = xy + yz + zx$. Ans: $v^2 = u + 2w$

3. If x, y, z are connected by a functional relation $f(x, y, z) = 0$, show that $\frac{\partial(y,z)}{\partial(x,z)} = \left(\frac{\partial y}{\partial x}\right)_{z=constt.}$.

4. If λ, μ, ν are the roots of the equation in k, $\frac{x}{a+k} + \frac{y}{b+k} + \frac{z}{c+k} = 1$, prove that $\frac{\partial(x,y,z)}{\partial(\lambda,\mu,\nu)} = -\frac{(\mu-\nu)(\nu-\lambda)(\lambda-\mu)}{(a-b)(b-c)(c-a)}$.

5. If $u = x(1 - r^2)^{\frac{-1}{2}}, v = y(1 - r^2)^{\frac{-1}{2}}, w = z(1 - r^2)^{\frac{-1}{2}}$, where $r^2 = x^2 + y^2 + z^2$ show that $\frac{\partial(u,v.w)}{\partial(x,y,z)} = (1 - r^2)^{\frac{-5}{2}}$.

6. If $u^3 = xyz, \frac{1}{v} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}, w^2 = x^2 + y^2 + z^2$, show that $\frac{\partial(u,v.w)}{\partial(x,y,z)} = -\frac{v(y-z)(z-x)(x-y)(x+y+z)}{3u^2\, w(xy+yz+zx)}$.

7. Find the Jacobian of $y_1, y_2, y_3, \ldots\ldots, y_n$ being given $y_1 = x_1(1 - x_2)$, $y_2 = x_1x_2(1 - x_3)$,....., $y_{n-1} = x_1x_2\ldots\ldots x_{n-1}(1 - x_n), y_n = x_1x_2\ldots\ldots x_n$ find $J(y_1, y_2, .., y_n)$

{**Hint:** $y_1+y_2+y_3, + \cdots .. + y_n = x_1$, Ans: $x_1^{n-1}x_2^{n-2} \ldots\ldots x_{n-1}$}.

8. Find the Jacobian $\frac{\partial(x,y,z)}{\partial(u,v.w)}$ being given $x = r\cos\theta \cos\phi, y = r \sin\theta\sqrt{1 - m^2\sin^2\phi}, z = r \sin\phi \sqrt{(1 - n^2\sin^2\theta)}$ where $m^2 + n^2 = 1$. {Hint:$r^2 = x^2 + y^2 + z^2$, Ans:$\frac{r^2(n^2\cos^2\theta + m^2\cos^2\phi)}{\sqrt{(1-n^2\sin^2\theta)(1-m^2\cos^2\phi)}}$}.

9. Prove that $JJ' = 1$

10. If $u = xyz, v = x + y - z, w = x + y + z$, Find $\frac{\partial(x,y,z)}{\partial(u,v.w)}$ Ans: $\frac{x}{2(2x^2-y^2)}$

11. If $f(0) = 0$ and $f'(x) = \frac{1}{1+x^2}$, prove without using the method of integration, that $f(x) + f(y) = f(\frac{x+y}{1-xy})$. {**Hint**: Let $u = f(x) + f(y)$ and $v = \frac{x+y}{1-xy}$ the find $J(u, v)$}

R.Y.Soni's

Tutorial-8, B. Tech. Sem-I,
(Triple integrals and their application)

Problem 1: Evaluate the triple integrals $\int_0^1 \int_1^{1-x} \int_0^{x+y} e^z \, dzdydx$ **Ans:** $\frac{1}{2}$

Problem 2: Evaluate $\iiint (x+y+z)^9 dxdydz$ over the region bounded by $x \geq 0, y \geq 0, z \geq 0, x+y+z \leq 1$. **Ans:** $\frac{1}{24}$

Problem 3: Evaluate the following triple integrals

(i) $\int_{-c}^{c} \int_{-b}^{b} \int_{-a}^{a} (x^2+y^2+z^2) \, dzdydx$ **(ii)** $\int_0^1 \int_{y^2}^1 \int_0^{1-x} x \, dzdxdy$

(iii) $\int_0^{\log 2} \int_0^x \int_0^{x+\log y} e^{x+y+z} \, dzdydx$

(iv) $\int_0^4 \int_0^{2\sqrt{z}} \int_0^{\sqrt{4z-x^2}} dydxdz$ **(v)** $\int_0^2 \int_0^2 \int_0^{4-x^2} (2x+y) dzdydx$

Ans: (i) $\frac{8abc(a^2+b^2+c^2)}{3}$**.(ii)**$\frac{4}{35}$ **(iii)**$\frac{8}{3}\log(2) - \frac{19}{9}$ **(iv)**8π **(v)** $\frac{80}{3}$

Problem 4: Evaluate the triple integral $\iiint_R (x-2y+z) \, dxdydz$, where R is the region determined by $0 \leq x \leq 1$, $0 \leq y \leq x^2$ and $0 \leq z \leq x+y$. **Ans:** $\frac{29}{105}$

Problem 5: Find the volume of the region bounded by the surface $y = x^2, x = y^2$ and the planes $z = 0, z = 3$. **Ans: 1**

Problem 6: Find the volume of the tetrahedron bounded by the co-ordinate planes and the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$. { Hint: $\int_0^a \int_0^{b(1-\frac{x}{a})} \int_0^{c(1-\frac{x}{a}-\frac{y}{b})} dzdydx$} **Ans:** $\frac{abc}{6}$

Problem 7: Find the volume common to the cylinder $x^2+y^2 = a^2$ and $x^2+z^2 = a^2$.

{ Hint: $\int_{-a}^{a} \int_{-\sqrt{(a^2-x^2)}}^{\sqrt{(a^2-x^2)}} \int_{-\sqrt{(a^2-x^2)}}^{\sqrt{(a^2-x^2)}} dzdydx$ } **Ans:** $\frac{16a^3}{3}$

Problem 8: Find the mass of the tetrahedron bounded by the co-ordinate planes and the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ the variable density $\rho = kxyz$

{ Hint: $\int_0^a \int_0^{b(1-\frac{x}{a})} \int_0^{c(1-\frac{x}{a}-\frac{y}{b})} kxyz \, dzdydx$ } **Ans:** $\frac{2ka^3}{3}(3\sqrt{3} - \frac{\pi}{3})$

Problem 9: Find the moment of inertia of the solid about its major axes generated by revolving the ellipse $\frac{a}{r} = \frac{2-\cos\theta}{2}$ about minor axes.

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Tutorial-9, B. Tech Sem-I
(Vector Calculus)

Note: In this exercise bold face letters (say) **F** represents vector $\vec{f}$ and i,j,k represents unit vectors $\hat{i},\hat{j},\hat{k}$ respectively.

Problem1: Evaluate the following (i) $\nabla\cdot(r^3\mathbf{r})$ (ii) $\nabla\cdot(r\,\nabla(\frac{1}{r^3}))$ (iii) $\nabla^2(\nabla\cdot(\mathbf{r}\,\frac{1}{r^2}))$(iv) grad Div($\frac{\mathbf{r}}{r}$) **Ans**: (i)$6r^3$, (ii)$3r^{-4}$, (iii)$2r^{-4}$, (iv) $-\frac{2\mathbf{r}}{r^3}$

Problem2: If **A**= $2yz\,i - x^2y\,j + xz^2k$, **B**= $x^2i + yz\,j - xy\,k$ and φ=$2x^2yz^3$, then find

(i) $(\boldsymbol{A}\cdot\varphi)\,\boldsymbol{B}$; **Ans:**

(ii) $\boldsymbol{A}\cdot\nabla\varphi$;

(iii) $(\boldsymbol{B}\cdot\nabla)\,\boldsymbol{A}$; (iv) $(\boldsymbol{A}\times\nabla)\varphi$; (v) $\boldsymbol{A}\times(\nabla\varphi)$

Problem3: If **A** and **B** are differentiable vector functions, ϕ and φ are differentiable scalar functions of position (x,y,z), then prove the following results:

(i)$\nabla\cdot(\boldsymbol{A}+\boldsymbol{B}) = \nabla\cdot\boldsymbol{A}+\nabla\cdot\boldsymbol{B}$

(ii) $\nabla\cdot(\varphi\boldsymbol{A}) = \varphi(\nabla\cdot\boldsymbol{A})+\boldsymbol{A}\cdot\boldsymbol{A}\times$

(iii) $\nabla\times(\varphi\boldsymbol{A}) = \varphi(\nabla\times\boldsymbol{A})+\boldsymbol{A}\times\nabla\varphi$

(iv) $\nabla\cdot(\boldsymbol{A}\times\boldsymbol{B}) = \boldsymbol{B}\cdot(\nabla\times\boldsymbol{A})-\boldsymbol{A}\cdot(\nabla\times\boldsymbol{B})$

(v) $\nabla(\boldsymbol{A}\cdot\boldsymbol{B}) = (\boldsymbol{B}\cdot\nabla)\boldsymbol{A}+(\boldsymbol{A}\cdot\nabla)\boldsymbol{B}+\boldsymbol{B}\times(\nabla\times\boldsymbol{A})+\boldsymbol{A}\times(\nabla\times\boldsymbol{B})$

(vi) $\nabla\times(\nabla\varphi)=0$

Problem4: Evaluate the grad. of $\log|\mathbf{r}|$ Ans: $\frac{\mathbf{r}}{r^2}$

Problem5: Show that $\nabla\varphi$ is a vector perpendicular to the surface $\varphi(x,y,z)$= constt.

Problem6: Find the directional derivative of $\varphi(x,y,z)$=$x^2yz+4xz^2$ at $(1,-2,-1)$ in the direction $2i-j-2k$. Ans:37/3

Problem7: Prove that the vector **A**= $3y^4z^2i + 4x^3z^2\,j - 3x^2y^2k$ is solenoidal.

Problem8: Prove following identities

(i) Div(∇φ×∇ψ)=0

(ii) If **A** and **B** are irrotational then **A**×**B** is solenoidal.

(iii) ∇×(φ ∇Φ)=0

(iv) Div (f ∇g) = f ∇^2g+∇f · ∇g

(v) $\mathbf{b}\cdot\nabla(\mathbf{a}\cdot\nabla\frac{1}{r}) = \frac{3(\mathbf{a}\cdot\mathbf{r})(\mathbf{b}\cdot\mathbf{r})}{r^5}-\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{r^3}$ where **a** and **b** are constt. Vectors.

(vi) $\nabla\cdot(U\nabla V - V\nabla U)$=U$\nabla^2U - V\,\nabla^2U$

(vii) $\nabla r^n = nr^{n-2}\boldsymbol{r}$

Problem9: (a) Prove that $\boldsymbol{F} = (y^2cosx+z^3)i+(2ysinx-4)j+(3xz^2+2)k$ is a conservative force field.

(b)Find the scalar potential for **F**. Ans: $(y^2sinx+xz^3-4y+2z+c)$

(c)Find the work done in moving an object in this field from $(0,1,-1)$ to $(\frac{\pi}{2},-1,2)$.

Ans: 15+4π

Problem10: Show that **V**=$2xyz\,i+(x^2z+2y)j+x^2yk$ is irrotational. Express **V** as gradient of a scalar function φ.

Problem11: Evaluate$\int_{(0,0)}^{(2,1)}(10\,x^4-2xy^3)dx-3x^2y^2dy$ along the path $x^4-6xy^3=4y^2$

Ans: 60. {Hint: Use Exact differential}

Problem12: Use Green's theorem to evaluate $\oint_C (cosx\,siny-xy)dx+sinx\,cosy\,dy$,where $C \equiv \mathbf{x^2+y^2=1}$. **Ans**: 0 .

Problem13: Verify Green's theorem in the plane for $\oint_C (xy+y^2)\,dx+x^2dy$,where C is the closed curve of the region bdd. by the line $y=x$ and curve $y=x^2$.

Problem14: Find the work done in moving a particle once around a circle C in the xy-plane, if circle has center at the origin and radius 3, Force field is given by $\boldsymbol{F} = (2x-y+z)i+(x+y-z^2)j+(3x-2y+4z)k$ Ans: 18π

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Problem15: State and prove Green's theorem.

Problem16: Prove that the area bounded by a simple closed curve C is given by $\frac{1}{2}\oint_C$ xdy − ydx{Hint: Use Green's theorem }

Problem16: Find the constants a,b,c such that **V**= $(x + 2y + az)i + (bx - 3y - z)j + (4x + cy + 2z)k$ is irrotational. Express V as gradient of a scalar function φ.

Problem18: State Green's theorem and hence evaluate $\oint_C$ (cosy)dx + (x − xsiny) dy ,where C is the closed curve $\mathbf{x^2 + y^2 = 1.}$

Problem19: Use Green's theorem to evaluate $\oint_C$ $(x^2 + xy)$dx + ($x^2 + y^2$) dy ,where C is the square formed by the lines $y = \pm 1, x = \pm 1$.

Problem20: Use Stoke's theorem to evaluate $\oint_C$ $(x + 2y)$dx + (x − z) dy + (y − z)dz ,where C is the boundary of the Δ with the vertices (2,0,0),(0,3,0),(0,0,6) oriented in the anticlockwise direction.

Problem21: Verify Stoke's theorem for **F**= x^2y k − y j + xzi and S is the surface of the region bounded by $x = 0, y = 0, z = 0, 2x + y + 2z = 8$, which is not included in the xz- plane .

Problem22: Evaluate $\iint_S (\nabla \times \mathbf{F}) \cdot \hat{n}$ ds ,where **F**= y i + (x − 2xz) j − xy k and S is the surface of the sphere $\mathbf{x^2 + y^2 + z^2 = a^2}$ above the xy-plane.
Ans: Zero.

Problem23: Evaluate $\iint_S$ $\mathbf{F} \cdot \hat{n}$ ds ,where **F**= 4xz i − y^2 j + yz k and S is the surface of the cube bounded by $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$. **Ans**: $\frac{3}{2}$.

Problem24: If **F**= (x^2 + y − 4)i + 3xy j + (2xz + z^2) k ,evaluate $\iint_S (\nabla \times \mathbf{F}) \cdot \hat{n}$ ds , and S is the surface of the sphere $\mathbf{x^2 + y^2 + z^2 = 16}$ above the xy-plane.
Ans: -16π

Problem25: Evaluate $\iint (y^2z^2 i + x^2z^2 j + x^2y^2 k) \cdot \hat{n}$ ds ,where S is the part of the surface of the sphere $\mathbf{x^2 + y^2 + z^2 = 1}$ above xy-plane.

Problem26: If V is the volume enclosed by the surface S, Find the value of $\int_S$ $\mathbf{r} \cdot \hat{n}$ ds
Ans: 3V

Problem27: State and prove Gauss Divergence theorm.

Problem28: Evaluate $\iint (ax\,i + by\,j + cz\,k) \cdot \hat{n}$ ds ,where S is the surface of the sphere $\mathbf{x^2 + y^2 + z^2 = 1.}$ [P.U. 2004]

Problem29: Evaluate $\iint_S$ $\mathbf{A} \cdot \hat{n}$ ds ,where **A**= 18z i − 12 j + 3y k and S is the part of the plane $2x + 3y + 6z = 12$ which is located in the first octant. **Ans**: 24

Problem30: Evaluate $\iint_S$ $\mathbf{A} \cdot \hat{n}$ ds ,where **A**= z i + x j − 3y^2z k and S is the surface of the cylinder $\mathbf{x^2 + y^2 = 16}$ included in the first octant between $z = 0$ and $z = 5$.
Ans: 90

Problem 31: Prove that (i) $\iint_S \frac{\mathbf{r}\cdot\hat{n}}{r^2}$ ds=$\iiint_V \frac{dv}{r^2}$; (ii) $\iint_S \frac{r^5\cdot\hat{n}}{}$ ds=$\iiint_V$ $5r^3\mathbf{r}$ dv

(iii) $\iint_S (\nabla \times \mathbf{F}) \cdot \hat{n}$ ds =0 for any closed surface S.

Problem 32: Use divergence theorem to evaluate $\iint_S$ $\mathbf{A} \cdot \overrightarrow{ds}$,where **A**= 4x i − 2y^2j + z^2k and S is the surface of the cylinder $\mathbf{x^2 + y^2 = 4}$ bdd. between $z = 0$ and $z = 3$.

Problem 33: Verify Stoke's theorem for **F**= (x^2 + y^2) i − 2xy j taken around rectangle bdd. by the lines $x = \pm a; y = 0, b$.

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